**Motion of charged particle in EM field**

**Charge in a magnetic field according to classical mechanics**

Consider an electron in a magnetic field **B** = B**k**. The motion of the electron in this field will be given by:



Splitting this up by components we’ll get:



We can take the derivative of the first equation w/r to time and plug it into the second equation…



The solution to this equation is of course:



Initial conditions demand that *a* must be v0x and so we get:



Using the first equation again, we get vy(t) to be:



Initial conditions demand that *b* = -v0y so we have:



and altogether, after integrating the z-equation we have:



Integrating these equations once to get the position as a function of time we have:



Initial conditions demand that:



So the solution is:



The motion described here is a circular in the x-y plane, with period T = 2π/ω = 2πm/|e|B, and of course constant velocity motion along the z-axis. This is evinced if we manipulate the equations into the following form:



Some interesting facts before we go. The radius of the orbit is clearly:



and the kinetic energy is:



**Example**

There is a ball centered at the origin of some coordinate system with a volume charge density 𝜌(𝑟)=𝜌0(𝑟2+𝛼𝑟3) for 𝑟<𝑅 and 𝜌(𝑟)=0 for 𝑟>𝑅.  𝑟 is the radial coordinate, 𝑅 is the radius of the ball, and both 𝜌0 and 𝛼 are positive numbers. A nearby particle has charge 𝑞<0, mass 𝑚, and is currently a distance 𝐿>𝑅 from the origin. What is the magnitude of the escape velocity for this particle (minimum velocity the particle needs to escape to an infinite distance from the ball)? Write the answer in terms of 𝜌0, 𝛼, 𝑅, 𝑞, 𝑚, 𝐿, and 𝜖0 (the vacuum permittivity). Assume there are no other charges present, and the only force we will consider is the electric interaction between the ball and the particle. Additionally, assume that the kinetic energy of the particle can be calculated non-relativistically. Neglect any energy loss due to radiation from acceleration of moving charges.

So first we need to the electric field of the ball,



where,



So,



We can get the potential via:



So the potential energy of the pair is:



The escape velocity required is given by energy conservation:



So yeah.

**Particle in Electric and Magnetic Field**

Consider a particle in an EM field, it’s motion will follow from:



We’ll presume constant fields, but arbitrary directions. So let’s take Laplace Transform of each side.



So we have:



where the tilde signifies Laplace transform. Now we’ll solve for the Laplace Transform. We can solve this by crossing and dotting both sides of (1) by **B**. We’ll cross first,



Using,



we have,



Filling this in,



Now dot both sides of (1) with **B**.



Now fill (2) into (1),



And fill (3) into this,



Now we can solve for .



And let’s use the cyclotron frequency,



to say,



solving for ,



we have:



So now ‘all’ we have to do is take the inverse Laplace Transform to get v(t). We could do this by hand, but I guess I’ll use Laplace Transform tables instead. Or maybe not. The inverse transform is:



The integral can be done using complex variables. It’s just the sum of the residues of the poles.



Do note the e in e2 is charge, not Euler’s number. Let’s do this term by term,



and then,



next,



Ugh. Well at least we kind of already did the next one previously.



Penultimate term. Kind of did that one too,



And final term,



So altogether, we have:



This isn’t the best way to write this though. We can combine part of the third term with the first term,



Now **v**0 – (**v**0·)is a vector whose magnitude is the component of **v**0 perpendicular to , and whose direction is pointing perpendicular to in the plane defined by **v**0×. If you think about it, we can write this direction as ×(0×). And **v**0×is a vector whose magnitude is also the component of **v**0 perpendicular to , but whose direction is perpendicular to the plane defined by **v**0×; the direction is in fact by 0× of course. And **v**0· is the component of **v**0 parallel to . These terms are simply a coordinate-free way to describe the results we found in the previous section – in the absence of an electric field, a particle will precess about the magnietic field, while also traveling at a constant speed along it. So I’ll write **v**(t) as:



Now can do similar manipulations on the E-terms. First I’ll combine parts of the last term with the fourth term,



and then recognize we have the component of E perpendicular to B, and the component parallel to B, and so write, by analogy with our previous work,



i.e.,



Well, recalling ωB = |e|B/m, we can write the last term as force over mass × time :



Okay, last time I’m going to regroup this:



Just kidding,



So this equation tells us three things. A charged particle will precess around the magnetic field with a velocity given by the vector sum of two separate precessional velocities with magnitudes v0⊥B and E⊥B/B (yes, the magnitudes of the vectors in the [ ]s is 1).



These precessional velocities aren’t necessarily in phase with each other. It will also accelerate along the magnetic field lines according to



And it will drift with a constant velocity,



If we’re in a 2D substance, then we can alternately formulate this drift result as saying that a charged particle will drift along the equipotentials of **E**, since **E**×**B** is perpendicular to **E** just as the equipotential would be. Interesting also that the direction of **v**drift is independent of the particle’s charge. Tried to illustrate these velocities below, for a negatively charged particle (well for small times, as it would eventually switch directions and go down the magnetic field lines):

A diagram of a cone with arrows and a cone

Description automatically generated

Note vdrift is coming out of the page.

**Example**

Last thing. This is an important special case that finds application in a lot of quantum mechanical discussions of the Hall effect. Say our charge starts from rest in perpendicular **E** =E, **B** = B. Then what will these formulas reduce to?



This looks like constant drift in the **j** direction with circular motion in the i-j plane superimposed upon it. So I guess it would go in a kind of curly cue motion.



So for a negative charge, it drifts in the direction of × with speed E/B, and simultaneously circulates around the magnetic field lines counter-clockwise with a period equal to the usual orbital frequency ωB = |e|B/m and speed E/B as well. So interestingly, there would be times when it is completely at rest. This would be when the charge is going around the circle in the direction opposite its drift velocity.